

## NASH RINGS ON PLANAR DOMAINS

BY

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**ABSTRACT.** Let  $D$  be a semialgebraic domain in  $R^2$ . Let  $N_D$  denote the Nash ring of algebraic analytic functions on  $D$ . Let  $A_D$  denote the ring of analytic functions on  $D$ . The main theorem of this paper implies that if  $\mathfrak{P}$  is a prime ideal of  $N_D$ , then  $\mathfrak{P}A_D$  is also prime. This result is proved by considering  $p(x, y)$  in  $R[x, y]$  and showing that  $p(x, y)$  can be put into a form so that its factorization in  $N_D$  is given by looking at its local factorization as a polynomial in  $y$  with coefficients which are analytic functions of  $x$ . Then for more general domains, a construction using the "complex square root" enables one to reduce to the case already considered.

Let  $D$  be a domain in  $R^2$ . Recall that a real valued function  $f: D \rightarrow R$  is a Nash function if (1)  $f$  is analytic at each point of  $D$ , i.e. expressible in convergent power series, and (2)  $f$  is algebraic, i.e. there exists a polynomial  $p_f(x, y, z)$  in  $R[x, y, z]$  such that  $p_f(x, y, f(x, y))$  vanishes on  $D$ . Let  $N_D$  = the ring of Nash functions on  $D$  and  $A_D$  = the ring of analytic functions on  $D$ . Let  $\mathcal{N}_D$  and  $\mathcal{A}_D$  be the corresponding sheaves. Then, if  $p(x, y) \in R[x, y]$ , the factorization of  $p(x, y)$  in  $A_D$  is known to be determined by its factorization in the completion of the local rings at each point  $P$  of  $D$ . This follows from the fact that  $H^1(U, \mathcal{A}_D^*) = 1$  for  $U$  open simply connected in  $D$ . But Hubbard [5] has shown that this cohomology group does not vanish when  $\mathcal{A}_D$  is replaced by  $\mathcal{N}_D$ . Recently Mostowski [6] proved (for any dimension actually) that for certain  $D$ ,  $p(x, y)$  factors in  $N_D$  at least as far as its disjoint components indicate. However, this is still far from showing that factorization is the same in  $N_D$  as in  $A_D$ . See the examples in §2, for instance.

In this paper, we prove that for  $D = R^2$ , the polynomial  $p(x, y)$  factors the same in  $N_{R^2}$  as in  $A_{R^2}$ . As a corollary, we deduce the result for any connected semialgebraic domain  $D$  and show directly that the class group of  $N_D$  is isomorphic to  $H^1(D, \mathcal{Z}_2)$ .

The proof of the result for  $R^2$  uses finding a suitable representation for  $p(x, y)$  in which its factorization as a polynomial in  $y$  determines its analytic and Nash factorizations. To get to more general domains, we analytically disconnect the zero set of  $p$  by using the transformation  $z \rightarrow \sqrt{z}$  to discon-

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nect at points outside  $D$ . That is, if  $D$  disconnects the zero set of  $p$ , then so will a finite number of points of  $R^2$  outside of  $D$ . By choosing the origin for our "square root" function at one of these points, we can transform  $R^2$  so the new polynomial will have its zeros disconnected and then we can apply the  $R^2$  result.

### 1. Factorization in $N_{R^2}$ .

**THEOREM 1.** *Let  $p(x, y) \in R[x, y]$ . Let  $N_{R^2}$  denote the Nash ring. Then  $p$  factors  $p = \prod q_i(x, y)$  where each  $q_i(x, y)$  is irreducible in  $N_{R^2}$ . Moreover, each  $q_i$  is also irreducible in  $A_{R^2}$ , the ring of analytic functions on  $R^2$ . Put another way, the irreducible factors of  $p(x, y)$  correspond to the analytically irreducible components of  $p = 0$  in  $R^2$ .*

**PROOF.** We wish to put  $p(x, y)$  in a form where we can factor it as a polynomial in  $y$  with coefficients which are polynomials in  $x$  so that when we look at the local factorizations, they extend to give the global factorizations. Such a form for  $p(x, y)$  will be achieved if  $p(x, y)$  has the following properties:

- (1.1)  $p(x, y)$  is monic in  $y$ .
- (1.2) All singular points  $(a, b)$  of  $p = 0$  with  $a$  real lie on the  $y$ -axis.
- (1.3) If  $p(a, b) = \partial p / \partial y(a, b) = 0$ ;  $a, b$  real;  $a \neq 0$ ; then  $\partial^2 p / \partial y^2(a, b) \neq 0$ .
- (1.4) If  $\partial p / \partial y(a, b) = 0, p(a, b) = 0, a \neq 0$  real; then  $b$  is real also.
- (1.5) Each connected component of  $V_R(p) - V_R(\partial p / \partial y)$  has closure which has nonempty intersection with the  $y$  axis.

Recall that  $V_R(p)$  denotes the real zeros of  $p$ .

The proof then consists of two parts. First, show that any  $p(x, y)$  can be put in the required form. Second, show that the conditions (1.1) to (1.5) are sufficient to factor  $p$  as we wish.

**LEMMA 1.** *Let  $q(y)$  be a rational function in  $R(y)$  with nonvanishing denominator (on  $R$ ). Then the transformation  $(x, y) \rightarrow (x + q(y), y)$  is a Nash isomorphism of  $R^2$  with itself.*

**PROOF.** Note that  $(x, y) \rightarrow (x - q(y), y)$  is the inverse map.

Transformations of the above type and those with the roles of  $x$  and  $y$  reversed are the only transformations we need to prove Theorem 1.

**LEMMA 2.** *Let  $P_i = (a_i, b_i), i = 1, \dots, n$ , be a finite number of points which in our case should include the singularities of  $V_R(p)$  with  $a_i$  real plus one point on each component of  $V_R(p)$  not already represented. Then there exists a pair of transformations of the allowed type so that the composition takes all  $P_i$  to the  $y$  axis.*

**PROOF.** First choose  $\lambda$  real  $\neq (b_j - b_i)/(a_i - b_j)$  for any  $i \neq j$ . Then take  $(x, y) \rightarrow (x, y + \lambda x)$  as the first transformation. Then we have  $b_i + \lambda a_i \neq b_j + \lambda a_j$  so that replacing  $P_i$  by its image, we can assume that  $b_i \neq b_j$  if  $i \neq j$ . Then there exists a monic polynomial  $q(y)$  such that  $q(b_i) = a_i$  for all  $i$ . By Lagrange interpolation, we can let

$$q(y) = \sum_i \left[ \prod_{j \neq i} (y - b_j) / \prod_{j \neq i} (b_i - b_j) \right] a_i$$

and note that if  $(a_i, b_i)$  is a singular point with  $a_i$  real, then so is  $(a_i, \bar{b}_i)$  where  $\bar{b}_i$  = the complex conjugate of  $b_i$ . From this it follows easily that  $q(y)$  has real coefficients. Then take  $(x, y) \rightarrow (x - q(y), y)$  as the second transformation and we are done.

We wish to show that we can put  $p(x, y)$  in the desired form. It is easy to achieve (1.1) by a transformation of the required type. Then we can apply Lemma 2 to achieve (1.2). Moreover, since (1.3) is a generic condition, we can take our first transformation in Lemma 2 so as to make the  $y$  coordinates where  $\partial p / \partial y = 0$  different from the  $b_i$  and then the second transformation can be chosen so (1.3) will be satisfied.

We now try for (1.5). The problem is that  $V_R(p)$  may have "dents" which do not cross the  $y$  axis. So we wish to push the dents in until they do cross the  $y$  axis. By making the new dent long enough we will have no problem with dents on the same side of the  $y$  axis. But the curve on the opposite side will be dented in the wrong direction so we first have to distort the curve so nothing will be opposite our proposed dents. This is done by a transformation of the form  $(x, y) \rightarrow (x, cx^{2n+1} + y)$ . We fix  $c > 0$  after some preliminary work, but  $n$  is chosen large enough so that the new  $p(x, y)$  will have the following  $M_1$  and  $M_2$  finite. So let

$$M_1 = \max\{b | \exists a, p(a, b) = 0, a \leq 0\},$$

$$M_2 = \min\{b | \exists a, p(a, b) = 0, a \geq 0\}.$$

Then choose  $c$  large enough so that if  $p(a, b) = 0$ ,  $\partial p / \partial y(a, b) = 0$  and  $a > 0$ , then  $b > M_1$ , and if  $p(a, b) = 0$ ,  $\partial p / \partial y(a, b) = 0$ , and  $a < 0$ , then  $b < M_2$ . This keeps the dents from being lined up opposite anything.

Let  $M = \max\{|b| | \exists a \text{ with } p(a, b) = 0, \partial p / \partial y(a, b) = 0\}$ , and  $(c_i, d_i)$  be the points where  $\partial p / \partial y(c_i, d_i) = 0$ ,  $p(c_i, d_i) = 0$ ,  $c_i \neq 0$ , and  $c_i \partial^2 p / \partial y^2(c_i, d_i) > 0$ . Now let  $\{(0, b_i)\}$  = the points where we have singular points and let  $q_i = M' \prod_{j=1}^s (y - b_j)^2 / (1 + N(y - d_1)^2)^\alpha$  where  $\alpha > s$  and where  $M'$  is chosen so that  $M' \prod_j (d_1 - b_j)^2 > M$ . We choose  $N$  so large that we can take the transformation  $(x, y) \rightarrow (x - q(y), y)$  and have one less component of  $V_R(p) - V_R(\partial p / \partial y)$  for the new  $p(x, y)$  which violates (1.5).

To see that this works, note that away from  $y = d_1$  the transformation is just  $(x, y) \rightarrow (x - \epsilon y, y)$  which does not add new critical points and we have

kept all the singular points on  $x = 0$  fixed. So we have to see what happens near  $y = d_1$ . So let  $d_1 = 0$ , and note  $q(y)$  will be like  $h(y)$  where  $h(y) = M/(1 + Ny^2)^\alpha$ . Then

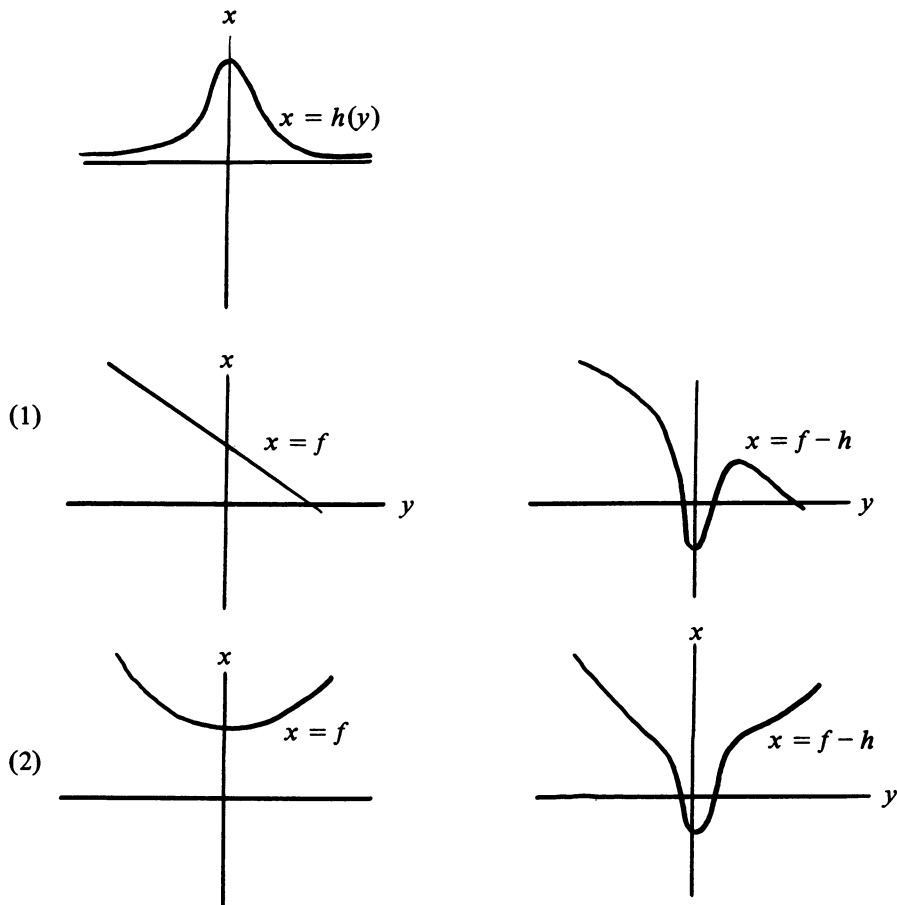
$$\dot{h}'(y) = -2MN\alpha y / (1 + Ny^2)^{\alpha+1}$$

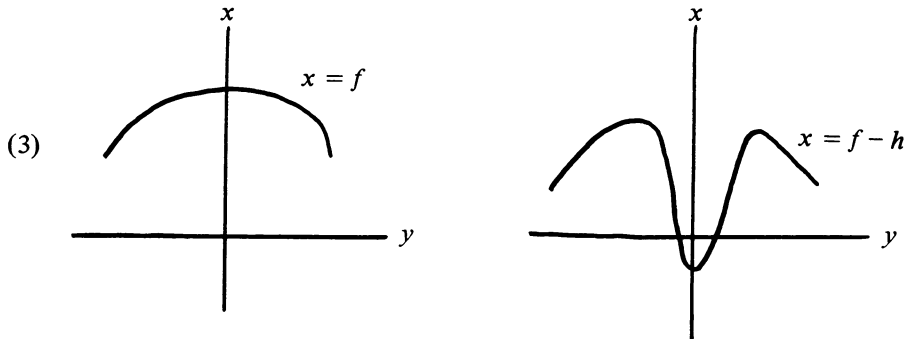
and

$$h''(y) = \alpha[-2MN(1 + Ny^2) + 4MN^2y^2(\alpha + 1)] / (1 + Ny^2)^{\alpha+2}.$$

Now for any constant  $\lambda$ , the equation  $h' = \lambda$  has at most two real roots for if  $g = (1 + Ny^2)^{\alpha+1} + 2MN\lambda y$ , then  $g'' > 0$  everywhere. We see that at each of the roots  $h''/h'$  is very large. For at one root of  $g$ , we have  $y \ll 1/\sqrt{N}$  so  $y \approx -1/2MN$  and  $h''/h' \approx 1/y$  and therefore is very large. At the other root of  $g$  we see  $y \approx N^{-\alpha/(2\alpha+1)}$  and  $h''/h' \sim N^{\alpha+1/(2\alpha+1)}$  which is also large for  $N$  large.

To illustrate what happens when we take our transformation, we sketch the possibilities when we subtract  $x = h(y)$  from  $x = f(y)$ .





In all three cases, we get no new dents which do not cross the  $y$ -axis and in case (2) we see we have made one of the old dents which did not cross the  $y$ -axis before, cross it in the new polynomial. By doing the same thing for the rest of the dents we can assume that we have achieved (1.5). To get a polynomial, we must, of course, clear denominators, at the end of each step.

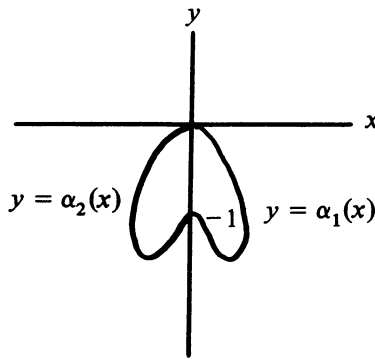
**LEMMA 3.** *Condition (1.4) is generic in the sense that for almost all  $\varepsilon$  the transformation  $(x, y) \rightarrow (x + \varepsilon y, y)$  will achieve it.*

**PROOF.** Take  $(a, b)$  with  $p(a, b) = 0$ ,  $\partial p / \partial y(a, b) = 0$ ,  $a$  real,  $b$  not real. Since  $\partial^2 p / \partial y^2(a, b) \neq 0$ , and  $\partial p / \partial x(a, b) \neq 0$ , as we are off the  $y$  axis,  $(a, b)$  is nonsingular. So we can expand  $x = \varphi(y)$  and get  $x - a = \sum_2^\infty \lambda_i (y - b)^i$  where  $\lambda_2 \neq 0$ . Now take the transformation  $(x, y) \rightarrow (x + \varepsilon y, y)$ , the new point where  $\partial p / \partial y = 0$  will no longer be real, since looking at the lowest degree terms of the new  $p = x + \varepsilon y - a - \lambda_2 (y - b)^2 + \dots$  we see that  $\partial p / \partial y = 0$  when  $y \approx b + \varepsilon / 2\lambda_2$ , and then at this point  $\text{Im } x \approx -\varepsilon \text{Im } b$  and so is  $\neq 0$ .

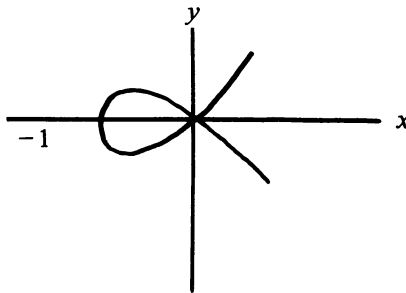
Now assume that  $p(x, y)$  satisfies (1.1)–(1.5); then we can use the Weierstrass Preparation Theorem [1, p. 74] to write  $p = \prod q_i$  with each  $q_i$  monic and analytic near  $x = 0$ . So each  $q_i = \prod (y - \alpha_{ij})$ . Therefore  $q_i$  is algebraic. Now  $y - \alpha_{ij}$  is analytic unless  $\alpha_{ij}$  is not, and  $\alpha_{ij}$  not analytic near  $x = a$  implies that  $\partial p / \partial y(a, \alpha_{ij}(a)) = 0$ . As we have already taken care of  $a = 0$ , we can assume  $a \neq 0$ . So we have if  $b = \alpha_{ij}(a)$ ,  $p(a, b) = \partial p / \partial y(a, b) = 0$ . Thus  $p(a, y)$  must have a multiple root and so we have for some  $i'j'$  that  $b = \alpha_{ij}(a) = \alpha_{i'j'}(a)$ . But by hypotheses,  $b$  is real and  $\partial^2 p / \partial y^2(a, b) \neq 0$  so near  $(a, b)$ , we have  $(y - \alpha_{ij})(y - \alpha_{i'j'})u_1 = p = u_2(y^2 + \lambda x + \mu xy + \dots)$  where  $u_1$  and  $u_2$  are locally units. But this clearly implies that  $(y - \alpha_{ij})(y - \alpha_{i'j'})$  is analytic near  $(a, b)$ . So if we multiply  $q_i$  times  $q_{i'}$  (unless  $i = i'$ ) we will get a function which is analytic near  $x = a$ . Thus when we have multiplied together all the factors  $q_i$  which we are forced to do by local conditions, we wind up with a global Nash function which corresponds to an analytic factor of  $p(x, y)$ .

## 2. Some examples.

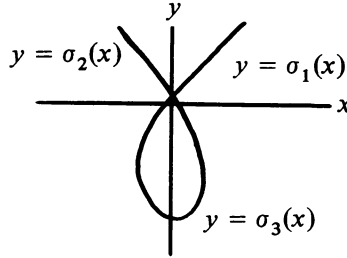
EXAMPLE 2.1 [2].  $p = x(x^2 + y^2) + x^4 + y^4$ . Then  $\partial p / \partial y = 2xy + 4y^3 = 0$  when  $y = 0$  or  $x = -2y^2$ . Solving, one sees that  $y = 0$  or  $y = \pm \sqrt{1 + \sqrt{2}} / 2$  and  $V_R(p)$  has a "dent". But interchanging the roles of  $x$  and  $y$ , one gets  $y(x^2 + y^2) + x^4 + y^4 = p$ . Then,  $\partial p / \partial y = x^2 + 3y^2 + 4y^3$ . One sees that the only places where  $p$  and  $\partial p / \partial y = 0$  simultaneously are when  $y = -1$ ,  $y = 0$  and  $y = -1 \pm \sqrt{3} / 4$ . Since these are all real numbers, condition (1.4) holds. Moreover the graph looks as below and therefore we get a factorization  $p = q_1 q_2$  where  $q_1 = (y - \alpha_1)(y - \alpha_2)$  and  $q_2$  has zero set equal to  $\{(0, 0)\}$ .



EXAMPLE 2.2. Let us consider the polynomial  $p(x, y) = y^4 - x^4 - x^5$ . Graph this and we get



Now note that  $p = (y^2 - x^2(1 + x)^{1/2})(y^2 + x^2(1 + x)^{1/2})$  which is good for  $x > -1$ . This factorization indicates that we should get a global factorization with one factor concentrated at  $(0, 0)$ . Our problem with  $p$  is that condition (1.3) is not satisfied since we see  $\partial p / \partial y = 4y^3$ ,  $\partial^2 p / \partial y^2 = 12y^2$  and both vanish at  $(-1, 0)$  which is on  $V_R(p)$ . However, as in the previous example, we interchange the roles of  $x$  and  $y$  and get  $p = y^5 + y^4 - x^4$ . Now graph this and get



So we get  $(y - \sigma_1)(y - \sigma_2)(y - \sigma_3)$  as one factor and the other factor is concentrated at the origin. The only problem might be with condition (1.4). But this can be checked as  $\partial p / \partial y = 5y^4 + 4y^3 = y^3(5y + 4)$  and so only is zero for real  $y$ .

**3. General planar semialgebraic domains.** Let  $D$  be a connected domain in  $R^2$  which is semialgebraic, i.e. defined by a finite number of polynomial inequalities. Then we wish to study the Nash ring  $N_D$  and show directly that  $N_D$  has class group isomorphic to  $H^1(D, \mathbb{Z})$ . It is already well known [4] that  $N_D$  is Noetherian and is clearly integrally closed and therefore is a Krull ring. Thus the divisor group is defined as in Bourbaki [3]. We know that if  $C$  is an analytically connected (in  $D$ ) component of  $V_R(p)$  for some polynomial  $p$ , then  $\mathfrak{P}_C = \{f \in N_D \mid f(C) = 0\}$  is a prime ideal. So we need only show that if  $C_1 \neq C_2$  are connected components of polynomials  $p_1, p_2$ , respectively, then  $\mathfrak{P}_{C_1} \neq \mathfrak{P}_{C_2}$ . It is enough to do this for  $C_1$  and  $C_2$  components of the same  $p$ . But while we are showing this, we also find out about the class group. Namely, we see that  $\mathfrak{P}_C^2 \sim (1)$  in the class group and if  $C_1$  and  $C_2$  are analytic arcs joining  $P_1$  to  $P_2$  where  $P_1$  and  $P_2$  are outside  $D$ , then  $\mathfrak{P}_{C_1}\mathfrak{P}_{C_2} \sim (1)$  in the class group. The proof uses first a Nash isomorphism sending  $P_2$  to infinity and then using the “square root” map, we find a 2-to-1 cover of  $R^2$  which is ramified only at  $P_1$ . We arrange things so that if  $\varphi: R^2 \rightarrow R^2$  is the cover, then  $\varphi^*C$  is analytically disconnected from the rest of the image of  $V_R(p)$ . Then by Theorem 1, there exists  $q(u, v)$  in  $N_{R^2}$  which vanishes only on  $\varphi^*C$  and generates  $I_{\varphi^*C}$  everywhere locally. Now  $q$  will not descend via  $\varphi$ , but  $q^2$  will in the sense that there exists a unit  $\beta$  in  $N_{R^2}$  so that if  $h = \beta q$ , then  $h$  descends to an element of  $N_{R^2}$  vanishing only on  $C$ .

**LEMMA 1.** *If  $C$  is an analytic arc of  $V_R(p)$  which has both ends at a point  $P$  not in  $D$ , then  $\mathfrak{P}_C$  is principal.*

**PROOF.** Send  $P = (a, b)$  to infinity by the transformation

$$\begin{aligned} x' &= (x - a) / [(x - a)^2 + (y - b)^2], \\ y' &= (y - b) / [(x - a)^2 + (y - b)^2]. \end{aligned}$$

This is a Nash isomorphism of  $R^2 - P$  with  $R^2 - (0, 0)$ . Now  $\varphi^*C$  is an analytic arc for which there exists, by Theorem 1, a Nash function  $h(x', y')$  which generates  $\mathfrak{P}_{\varphi^*C}$ , and pulling  $h$  back we get the required generator of  $\mathfrak{P}_C$ .

LEMMA 2. *Let  $C$  be an analytic arc of  $V_R(p)$  running from  $P_1$  to  $P_2$  where  $P_1$  and  $P_2$  are both outside  $D$ . Then there exists  $h$  in  $N_D$  such that  $h$  vanishes in  $D$  only on  $C$ .*

PROOF. We send  $P_2$  to infinity as in Lemma 1. Then we can certainly move  $P_1$  to  $(0, 0)$  for convenience. We first want to look at  $V_R(p)$  at  $(0, 0)$ . If  $C$  is part of a cusp at this point, we want the angle of the cusp to be  $180^\circ$  not  $0^\circ$ . The principal term of the cusp can be assumed to be  $y^m - x^n$  where  $m < n$ . When  $m$  is odd and  $n$  is even, the cusp will already have angle  $180^\circ$ , and if  $m$  and  $n$  are even, then we get a factorization so that  $C$  will be part of only one branch of the cusp so we can assume that  $m$  is even and  $n$  is odd. But consider the transformation defined by  $x = s$ ,  $y = t(x^2 + t^2)^\alpha$  where  $\alpha$  is a large integer. Then if  $\psi$  is the corresponding transformation,  $\psi$  is a one-to-one Nash transformation of  $R^2 - (0, 0)$  onto itself. To see this, note, solving  $(t^2 + x^2)^\alpha t - y = 0$  for  $t = f(x, y)$ , we see that  $f$  is a Nash function on  $R^2 - (0, 0)$ . For if  $g(t) = t(t^2 + x^2)^\alpha - y$ , then  $g'(t) = (t^2 + x^2)^\alpha + 2\alpha t^2(t^2 + x^2)^{\alpha-1} > 0$  unless  $t^2 + x^2 = 0$ , i.e.  $x = y = 0$ . The cusp of  $\varphi^*C$  has Newton polygon, the line joining  $(0, n)$  to  $((2\alpha + 1)m, 0)$ , so the dominant term is  $s^n - t^{(2\alpha+1)m}$  which will have  $180^\circ$  angle for  $m$  large enough.

Now consider the map  $\varphi: R^2 \rightarrow R^2$  defined by  $x = u^2 - v^2$ ,  $y = 2uv$  (i.e. just " $z \rightarrow \sqrt{z}$ "). Then  $\varphi$  will define a two-to-two cover of  $R^2$  by  $R^2$  ramified only at  $(0, 0)$ . Moreover, since the angle of the branch of  $C$  through  $(0, 0)$  is  $180^\circ$ , we see that  $\varphi^*C$  is analytically disconnected from  $\varphi^*C'$ , where  $C' =$  the rest of  $V_R(p)$ . By the form of  $\varphi$  we see that  $(a, b)$  in  $\varphi^*V_R(p)$  iff  $(-a, -b)$  is in  $\varphi^*V_R(p)$ . But since the angle between  $\varphi^*C$  and  $\varphi^*C'$  will be  $90^\circ$ ,  $\varphi^*C$  will also be invariant under  $(u, v) \rightarrow (-u, -v)$ . Thus  $q(-u, -v)$  will also vanish on  $\varphi^*C$ . But since by Theorem 1,  $q$  locally generates the zero set of  $\varphi^*C$ , we see that  $q(-u, -v) = \lambda(u, v)q(u, v)$ . This also means that  $q(u, v)$  vanishes to degree 1 over most of  $\varphi^*C$ .

Now  $q(u, v) = \lambda(-u, -v)q(-u, -v) = \lambda(u, v)\lambda(-u, -v)q(u, v)$ , which implies  $\lambda(u, v)\lambda(-u, -v) = 1$ , i.e.  $\lambda$  is a unit in  $N_{R^2}$ . So Lemma 2 will follow from Lemma 3.

LEMMA 3. *Assume  $u^2 - v^2 = x$ ,  $2uv = y$ , as above, and  $q(u, v)$  is in  $N_{R^2}$  with  $q(u, v) = \lambda(u, v)q(-u, v)$  with  $\lambda$  a unit of  $N_{R^2}$ . Then let  $\lambda = \pm\beta^2$ . If  $h = q\beta^{-1}$  we find that  $h(u, v) = \pm h(-u, -v)$  and so  $h$  descends to the  $x$ - $y$  plane if  $\lambda > 0$ .*



PROOF. Since  $h(u, v) = \beta^{-1}(u, v)q(u, v)$  we see that  $h(-u, -v) = q(-u, -v)\beta^{-1}(-u, -v)$ . Therefore  $q(u, v) = \pm\beta^2(u, v)q(-u, -v) = \beta^2(u, v)\beta^2(-u, -v)q(u, v)$ . Then  $\beta^2(u, v)\beta^2(-u, -v) = 1$  which implies  $\beta(u, v)\beta(-u, -v) = 1$  (since  $\beta > 0$ ). Finally,

$$\begin{aligned} h(u, v) &= q(u, v)\beta^{-1}(u, v) = \pm\beta^2(u, v)q(-u, -v)\beta^{-1}(u, v) \\ &= \pm\beta(u, v)q(-u, -v) = \pm q(-u, -v)\beta^{-1}(-u, -v) \\ &= \pm h(-u, -v). \end{aligned}$$

So Lemma 3 follows from Lemma 4.

LEMMA 4. If  $h(u, v) = h(-u, -v)$ , and  $h$  is in  $N_{R^2}$ , then  $h$  descends via  $\varphi$  to  $N_{R^2} - (0, 0)$ .

PROOF. Let  $(a, b) \neq (0, 0)$  in the  $u$ - $v$  plane. Then  $(a, b)$  and  $(-a, -b)$  are the two points identified by  $\varphi$ . So it is obvious that  $h$  descends to a function on the  $x$ - $y$  plane minus  $(0, 0)$ . That  $h$  descends to a Nash function is also clear since  $\varphi$  is analytic outside  $(0, 0)$ . Moreover, if  $p(z, u, v) = \sum a_i(u, v)z^i$  is the irreducible polynomial for  $h$  over  $R[u, v]$  (localized by all polynomials not vanishing on  $R^2$ ), then since  $\sum a_i(u, v)h(u, v)^i = 0 = \sum a_i(-u, -v)h(u, v)^i$ , we see all  $a_i(u, v) = a_i(-u, -v)$  and so descend; in fact, one can see that  $a_i$  is a rational function of  $u^2$ ,  $uv$  and  $v^2$ . But solving explicitly one finds  $u^2 = (x + \sqrt{x^2 + y^2})/2$ ,  $v^2 = (-x + \sqrt{x^2 + y^2})/2$  and  $2uv = y$ .

To finish the proof of Lemma 2, note that while the  $q$  constructed there is such that  $q(u, v) = \lambda(u, v)q(-u, -v)$  where  $\lambda < 0$ ,  $q^2(u, v) = \lambda^2(u, v)q^2(-u, -v)$  so the  $h$  constructed from  $q^2(u, v)$  will descend and vanish only on  $C$ .

THEOREM 2. Let  $D$  be a connected domain in  $R^2$  which is semialgebraic. Then if  $C$  is an analytically connected arc of some  $V_R(p)$  with  $C$  as large as possible, then we have  $\mathfrak{P}_C = \{f | f(C) = 0 \text{ is a prime ideal of } N_D\}$ . If  $C_1 \neq C_2$ , then  $\mathfrak{P}_{C_1} \neq \mathfrak{P}_{C_2}$ . Moreover, the class group of  $N_D$  is isomorphic to  $H^1(D, \mathbb{Z}_2)$ .

PROOF. The first part of the theorem follows from Lemma 2. For the rest we prove some more lemmas.

LEMMA 5. Let  $P_1, P_2$  be points of the complement of  $D$  and let  $C_1$  and  $C_2$  be analytically connected arcs running from  $P_1$  to  $P_2$ . Then if  $I_C$  = the ideal of Nash functions of  $D$  vanishing on  $C$ , we have that  $I_{C_1}I_{C_2} \sim (1)$  in the class group.

PROOF. As in the proof of Lemma 2, we can construct in the two-to-one covering of  $R^2$ , Nash functions  $q_1(u, v)$ ,  $q_2(u, v)$  vanishing on  $\varphi^*C_1$ ,  $\varphi^*C_2$ , respectively. Then for the corresponding  $h_i(u, v)$ , we have  $h_i(u, v) = -h_i(-u, -v)$ . So  $h_1h_2(u, v) = h_1h_2(-u, -v)$  and descends to a Nash

function on  $R^2 - (0, 0)$ . So this function will generate  $I_{C_1}I_{C_2}$ .

**LEMMA 6.** *Consider  $P_1, P_2, P_3$  three points in the complement of  $D$ . Let  $C_1, C_2$ , and  $C_3$  be analytically connected algebraic arcs of  $D$  where  $C_1$  joins  $P_1$  and  $P_2$ ,  $C_2$  joins  $P_2$  to  $P_3$  and  $C_3$  joins  $P_1$  to  $P_3$ . Then  $\mathfrak{P}_{C_1}\mathfrak{P}_{C_2} \sim \mathfrak{P}_{C_3}$  in the class group.*

**PROOF.** There exists an analytic arc of an algebraic curve passing through  $P_1, P_2$  and  $P_3$  in order. In fact, some parabola will usually work. Then if  $C'_3$  denotes this arc and  $C'_1$  denotes the piece from  $P_1$  to  $P_2$  and  $C'_2$  denotes the piece between  $P_2$  and  $P_3$ ; let  $I_C$  = the ideal of Nash functions on  $D$  vanishing on  $C$ . Then by Lemma 5,  $I_{C'_1}I_{C'_2} \sim I_{C'_3}$  and  $\mathfrak{P}_{C'_i} \sim (1)$ ,  $i = 1, 2, 3$ . So the result follows.

**LEMMA 7.** *Choose a point  $P_i$  in each hole  $H_i$  of  $D$ . Then if  $C$  is an analytically connected algebraic arc ending in  $H_i$ , there exists an arc  $C'$  with the same other end as  $C$  so that  $I_{C'} \sim I_C$  and  $C'$  ends at  $P_i$  in  $H_i$ .*

**PROOF.** We can join  $P$ , the end of  $C$  in  $H_i$  to  $P_i$  by a finite number of analytic arcs of algebraic curves. Let these arcs be  $L_1, \dots, L_r$ . All of these arcs are in  $H_i$  so not in  $D$ . So  $I_{C'} \sim I_C I_{L_1} \cdots I_{L_r}$ . This is because each  $I_{L_i} \sim (1)$  in the class group of  $N_D$  since  $L_i \cap D = \emptyset$ . Now use Lemma 6 and induction.

**COROLLARY.** *If  $\prod I_{C_i} = I$  is an ideal of  $N_D$  where each  $C_i$  runs from one hole of  $D$  to another, then  $I$  is principal iff the number of ends at each hole is even. This shows directly that the class group of  $N_D \cong H^1(D, \mathbb{Z}_2)$ .*

**PROOF.** This follows from the preceding lemmas.

#### 4. Some examples pertaining to §3.

**EXAMPLE 4.1.** Let  $D = R^2 - (0, 0)$ . Let  $p = y$  so  $V_R(p) =$  the  $x$ -axis. Now  $V_R(p)$  is disconnected in  $D$ . So our cover is given by  $x = u^2 - v^2, y = 2uv$ . If  $\varphi$  is the covering map, then  $\varphi^*p = 2uv$ . And since the map  $\varphi^{-1}$  is the "square root map", we see that the positive  $x$ -axis corresponds to  $v = 0$ , and the negative  $x$ -axis to  $u = 0$ . As in §3,  $u$  will not descend but  $u^2$  will and  $u^2 = (x + (x^2 + y^2)^{1/2})/2$ , vanishing on the positive  $x$ -axis.

**EXAMPLE 4.2.** Let  $D$  be as in Example 4.2. Let  $\mathfrak{P}_1 = \{f \in A_D \text{ such that } f \text{ vanishes on the positive } x\text{-axis}\}$ . Let  $\mathfrak{P}_2 = \{f \in A_D \text{ such that } f \text{ vanishes on the negative } y\text{-axis}\}$ . The  $\mathfrak{P}_1\mathfrak{P}_2$  should be principal. Taking our cover as in the previous example, we find that the function we want to descend is  $(u + v)v = uv - v^2$  which descends to  $y/2 + (x - (x^2 + y^2)^{1/2})/2$  which does vanish to degree one on the zero set of  $\mathfrak{P}_1\mathfrak{P}_2$  in  $D$ .

Of course, these examples are rather trivial but for that reason they make the general theory more transparent.

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